# GROUP-THEORETIC CONSIDERATIONS LEAD TO NEW SOLUTIONS OF THE WATER WAVE PROBLEM

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## Introduction

The classical problem of the irrotational, two-dimensional motion of an inviscid, incompressible fluid, bounded below by a rigid horizontal plane and above by a free surface, was recently reanalyzed by Bridges & Dias (1989), from a group-theoretic point of view. The effects of both gravity and surface tension were included. This new approach, which makes use of the Hamiltonian structure and of the fundamental symmetries of capillary-gravity waves: translational invariance in $x$ (horizontal direction), reflectional invariance in $x$, and translational invariance in $t$ (time), has already led to interesting and sometimes quite surprising physical and mathematical results:
(a) for an open region of $(\delta, \sigma)$ parameter space ($\delta$ is the mean depth and $\sigma$ the surface tension coefficient), the travelling wave (TW) and standing wave (SW) branches of solutions are connected by a new branch of solutions,
(b) if a SW is perturbed by a small horizontal impulse, it disintegrates into a torus (quasi-periodic solution). In fact, in the energy-momentum space, the SW and TW solutions are surrounded by tori.
(c) the $\mathbf{O}(2)$ symmetry provides an interesting link between waves in a plane, waves in a circular basin and the spherical pendulum,
(d) symmetry breaking perturbations, such as the introduction of a current, the reduction to a finite domain, lead to more complex dynamics because the reduced Hamiltonian is no longer integrable,
(e) by analogy with the work of Miles on the spherical pendulum, a harmonic ambient pressure field at the free surface leads to a period doubling cascade and ultimately a chaotic motion.

In view of this successful group-theoretic approach to nonlinear water waves, we have also reanalyzed the $(m,n)$ mode-interaction problem for capillary-gravity waves (see Chen & Saffman (1979)). Chen & Saffman showed that in deep water waves characterized by a resonant interaction between the $m$th and the $n$th harmonics exist for all sets of naturals $m$ and $n$ for appropriate values of the Bond number $B$, which is defined as

$$B = \frac{\rho g L^2}{4\pi^2 \sigma} \tag{1}$$

where $\rho g$ is the weight density of water and $L$ the wave length. Without loss of generality, $m$ and $n$ are taken to be relatively prime. It can be shown that there are three distinct cases: $(m,n) = (1,2)$, $(m,n) = (1,3)$ and $m + n > 4$. When $(m,n) = (1,2)$, the interaction is quadratic. Otherwise, it is cubic.

In the first part, the governing equations and symmetries of the water wave problem are briefly presented. In the second part, the new branch of solution connecting travelling and standing waves is presented. Then, Wilton's ripples, which correspond to $m = 1$ and $n = 2$ are reanalyzed from a group-theoretic point of view. New classes of waves are obtained. Next, the case $m = 1$, $n = 3$ is considered. Again, new solutions are obtained. The physical aspects of the unusual class of these new waves (three-mode mixed waves) are illustrated by plotting the profile of the free surface as a function of $x$ for discrete values of $t$. In the last part, the analysis is continued by studying $(m, n)$ mode-interaction with $m + n > 4$.

## Governing equations and symmetries

Two-dimensional irrotational water waves in a laterally unbounded domain of constant depth $\delta$ are considered. Taking as variables to describe the elevation of the free surface $\eta(x,t)$ and the velocity potential evaluated at the free surface $\Phi(x,t)$, the evolution equations for the system can be written with the following Hamiltonian structure:

$$\eta_t = D_\Phi H \quad , \quad \Phi_t = -D_\eta H \tag{2}$$

where $H$ is the total energy of the waves and $D$ represents a functional derivative in the sense of the calculus of variations. For capillary-gravity waves,

$$H = \int_0^L \{\frac{1}{2}\Phi R \Phi_{(\mathbf{n})} + \frac{1}{2}\eta^2 + \sigma(R-1)\}dx \tag{3}$$

where $R = (1 + \eta_x^2)^{1/2}$ and $\mathbf{n}$ is the unit normal directed outwards.

The continuous symmetries for two-dimensional capillary-gravity waves were studied in detail by Benjamin & Olver (1982). Two of these play a fundamental role: the translational invariance in $x$ and the translational invariance in $t$. Moreover, the discrete symmetry associated with the reflectional invariance in $x$ is also essential. It is the group $\mathbf{O}(2) \times \mathbf{S}^1$, generated by these three symmetries, which forms the basis of our analysis.

## Waves without mode interactions

With the $\mathbf{S}^1$ symmetry in time and the $\mathbf{SO}(2)$ symmetry in space, the problem is posed on a set of doubly periodic functions. The linearized water wave problem is then associated with the four dimensional space $\{\cos t, \sin t\} \times \{\cos x, \sin x\}$ in the absence of mode interactions. Associating complex coordinates $(z_1, z_2)$ with this space, the linearized problem is spanned by two travelling waves

$$\eta(x,t) = 2\mathrm{Re}\{z_1 e^{i(\omega t - kx)} + z_2 e^{i(\omega t + kx)}\}. \tag{4}$$

The normal form, i.e. the relationship between the natural frequency and the coordinates $z_1$ and $z_2$, is obtained to third order by assuming perturbation expansions of $\eta$, $\phi$ and $\omega$ in powers of a small parameter $\epsilon$, measure of the wave amplitude. An analysis of the normal form shows that there are three non-trivial classes of periodic solutions: the usual travelling waves ($z_1 = 0$ or $z_2 = 0$) and standing waves ($z_1 = z_2$), but also Z-waves for which $z_1 \neq z_2$. These waves were first observed by Bridges & Dias (1989).

### Wilton's ripples

For Wilton's ripples, characterized by an interaction between the first and the second harmonic, the linearized problem becomes associated with the eight dimensional space $\{\cos t, \sin t\} \times \{\cos x, \sin x\} \oplus \{\cos 2t, \sin 2t\} \times \{\cos 2x, \sin 2x\}$. Associating complex coordinates $(z_1, z_2, z_3, z_4)$ with this space, the linearized problem is spanned by four travelling waves

$$\eta(x,t) = 2\mathrm{Re}\{z_1 e^{i(\omega t - kx)} + z_2 e^{i(\omega t + kx)} + z_3 e^{2i(\omega t - kx)} + z_4 e^{2i(\omega t + kx)}\} \tag{5}$$

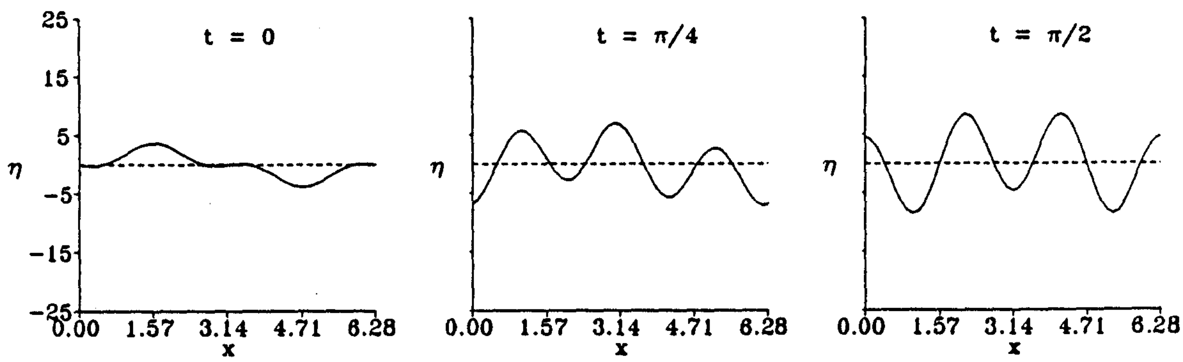
The normal form is obtained to third order by a perturbation method. An analysis of the normal form shows that there are seven non-trivial classes of periodic solutions: 2-crested SW, 2-crested TW, combination 2-crested and 1-crested TW (the classical Wilton's ripple), combination 2-crested and 1-crested SW (studied by Vanden-Broeck (1984)), 3-mode mixed waves, a second family of SW and a secondary branch of Z-waves. The analysis is still under way.

### Third-harmonic resonance for capillary-gravity waves

When there is resonance between the first and the third harmonic, the linearized problem becomes associated with the eight dimensional space $\{\cos t, \sin t\} \times \{\cos x, \sin x\} \oplus \{\cos 3t, \sin 3t\} \times \{\cos 3x, \sin 3x\}$. Associating complex coordinates $(z_1, z_2, z_3, z_4)$ with this space, the linearized problem is spanned by four travelling waves

$$\eta(x,t) = 2\mathrm{Re}\{z_1 e^{i(\omega t - kx)} + z_2 e^{i(\omega t + kx)} + z_3 e^{3i(\omega t - kx)} + z_4 e^{3i(\omega t + kx)}\} \tag{6}$$

The normal form is obtained to third order by a perturbation method. An analysis of the normal form shows that there are six non-trivial classes of periodic solutions: 3-crested SW, 3-crested TW, combination 3-crested and 1-crested TW, combination 3-crested and 1-crested SW, a secondary branch of Z-waves and mixed waves. The mixed waves were first observed by Dias & Bridges (1989). They are unusual and unexpected. They are a combination of two right-running waves with different amplitude and one left-running wave. The physical form of a typical 3-mode mixed wave is shown below:



### $(m, n)$ mode-interaction for capillary-gravity waves

The method used for the (1,2) and (1,3) mode-interactions can be generalized to the $(m, n)$ mode-interaction problem for capillary- gravity waves. The analysis was done recently by Bridges(1989). A classification theorem for group-invariant Hamiltonian systems can be used to show that there are between six and thirteen non-trivial classes of periodic solutions in O(2)-invariant Hamiltonian systems with a mode-interaction.

### Conclusion

The approach used to reanalyze the classical water wave problem shows the importance of the role of symmetry. We have good reasons to believe that this approach could be used to reanalyze more complicated wave-related problems (floating bodies, ship waves, internal waves).

### References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## DISCUSSION

Miloh: I am sure that out of the fourteen different solutions found most of then will appear as unstable modes or should be ruled out based on physical grounds. Have you looked into this possibility?

Bridges & Dias: First, the number of classes of solutions depends on the values of (m,n). There are between 8 and 14 classes. In order to study existence questions for each of the classes of periodic solutions (we have not done it), one could follow the analysis of Toland and Jones (1985) for studying (m,n) travelling waves, and work in a function space with the symmetry of the particular class. I think that all the classes are physically acceptable.