SHORT – WAVE ASYMPTOTICS IN TWO DIMENSIONAL WATER WAVE PROBLEMS.

by

P. WALTON.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANCHESTER
MANCHESTER, M13 9PL, ENGLAND.

Problems in short-wave asymptotics present great difficulty in the linearized theory of water waves. There are very few theoretical results, even for specialized bodies, and those which have been obtained give only a few leading terms and are not good enough for many applications. Results in short-wave asymptotics are of particular importance since numerical schemes based on methods such as integral equations become increasingly inaccurate and expensive as the wavelengths in the problem become shorter. Short-wave results are needed in problems such as time-dependent wave problems and the slowly varying force where the potential needs to be known throughout the frequency range. In this paper we shall present a new method for calculating short-wave asymptotics in linear water wave theory. This method involves the explicit construction of a Green function with uniform short-wave asymptotics and should lead to theoretical results for specialized bodies as well as to a numerical scheme for more arbitary bodies. For bodies which meet the free surface at right-angles this Green function is formed by introducing an extra source potential inside the body near the free surface. For other angles of intersection we also need a 'weak' dipole inside the body. The position of this source and dipole is chosen so that the Green function has small normal derivative near the points of intersection of the body and free surface. We will begin by illustrating how we use such a Green function.For simplicity we shall assume that we have a single body with surface S and that the body meets the free surface at right angles.

Let us apply Green's theorem to a suitably chosen Green function and the velocity potential ϕ in order to obtain an integral equation for the values of ϕ on S:

$$\pi\phi(p) + \int_S \phi(q)K(p,q)\,dS_q = \int_S \frac{\partial\phi}{\partial n_q}(q)G(p,q)\,dS_q \quad ,p \in S \tag{1}$$

where $G(p,q)$ depends on the wave number, $K(p,q) = \frac{\partial G}{\partial n_q}(p,q)$ and $\frac{\partial}{\partial n_q}$ denotes normal differentiation acting into the fluid. We now suppose that the Green function G has been chosen so that both $G(p,q)$ and $K(p,q)$ have explicit uniform asymptotics of the form

$$G(p,q) \sim G_W(p,q;N) + \sum_{n=0}^{\infty} \frac{G_n(p,q)}{N^n} \tag{2}$$

$$K(p,q) \sim K_W(p,q;N) + \sum_{n=0}^{\infty} \frac{K_n(p,q)}{N^n} \tag{3}$$

where G_W and K_W are wave terms and N is a large parameter associated with the short-wave limit (for example the wave number). Using an obvious operator notation we may write (1) in the form

$$\left[\tilde{I} + \frac{\tilde{K}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{K}_n}{N^n}\right]\phi \sim \left[\frac{\tilde{G}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{G}_n}{N^n}\right]\frac{\partial\phi}{\partial n_q} \tag{4}$$

We may use Watson's lemma to find the asymptotics of the $\tilde{G}_W$ term in (4) and thus we have

$$\left[\tilde{I} + \frac{\tilde{K}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{K}_n}{N^n}\right]\phi \sim \sum_{n=0}^{\infty}\frac{f_n^{(0)}}{N^n} \tag{5}$$

where $f_n^{(0)}$, $n = 0, 1, \ldots$ are known functions and are all $o(N)$. We shall now use (5) to propose the following method of obtaining asymptotics for ϕ: Suppose that by either theoretical or numerical means we may solve the infinite frequency problem given by

$$(\pi\tilde{I} + \tilde{K}_0) = f_0^{(0)} \ (\equiv \tilde{G}_0\frac{\partial\phi}{\partial n_q}) \tag{6}$$

then by setting $\phi = \phi_0 + \psi_1$ we may re-write (5) as

$$\left[\tilde{I} + \frac{\tilde{K}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{K}_n}{N^n}\right]\psi_1 \sim \sum_{n=1}^{\infty}\frac{f_n^{(0)}}{N^n} - \frac{\tilde{K}_W}{\pi}\phi_0 - \sum_{n=1}^{\infty}\frac{\tilde{K}_n}{N^n}\phi_0. \tag{7}$$

Using Watson's lemma to examine the $\tilde{K}_W$ term on the right-hand side of (7) we may see that provided the Fredholm determinant is not small $\psi_1 = o(1)$ for large N. We can be assured that the Fredholm determinant is not small since our Green function is constructed so as to have a small normal derivative near the points of intersection of the body and the free surface. We may now repeat this procedure: We may use Watsons lemma to re-write (7) as

$$\left[\tilde{I} + \frac{\tilde{K}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{K}_n}{N^n}\right]\psi_1 \sim \sum_{n=1}^{\infty}\frac{f_n^{(1)}}{N^n} \tag{8}$$

where once again $f_n^{(1)}$, $n = 1, 2, \ldots$ are known functions and are all $o(N)$. We may now write $\psi_1 = \phi_1/N + \psi_2$ where ϕ_1 is the solution of

$$(\pi\tilde{I} + \tilde{K}_0)\psi_1 = f_1^{(1)}$$

so that

$$\left[\tilde{I} + \frac{\tilde{K}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{K}_n}{N^n}\right]\psi_2 \sim \sum_{n=2}^{\infty}\frac{f_n^{(1)}}{N^n} - \frac{\tilde{K}_W}{\pi}\frac{\phi_1}{N} - \sum_{n=1}^{\infty}\frac{\tilde{K}_n}{N^n}\frac{\phi_1}{N}. \tag{9}$$

and we may once again define $f_n^{(2)}$ $n = 2, 3, \ldots.$ which are all $o(N)$. Continuing in this way we find that $\phi = \sum_{n=0}^{M}\phi_n/N^n + \psi_{M+1}$ where

$$\left[\tilde{I} + \frac{\tilde{K}_W}{\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\tilde{K}_n}{N^n}\right]\psi_{M+1} \sim \sum_{n=M+1}^{\infty}\frac{f_n^{(M+1)}}{N^n} \tag{10}$$

with f_n^{M+1} all $o(N)$ so that $\psi_{M+1} = o(1/N^M)$ and we have an asymptotic expansion. [In this procedure Watson's lemma and the $f_n^{(m)}$ notation was used to show that the expansion resulting from our procedure was on asymptotic one; in any numerical scheme based on this procedure the wave term would be evaluated numerically.]

It is not yet clear how successful this method will prove, although in principle it should lead to the complete asymptotic expansion in a given problem.