NONLINEAR STANDING WAVES IN A TWO-DIMENSIONAL HEAVING TANK

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Introduction

The subharmonic standing waves in a closed basin forced to oscillate vertically was first observed by Faraday (1831), and later explained by Benjamin & Ursell(1954) as a form of parametric resonance whose response amplitudes are governed by a Mathieu's equation. Ockendon & Ockenden (1973) applied multiple-scale analysis to obtain an evolution equation governing the response near sub-harmonic resonant frequencies. Unfortunately, they did not give explicit forms of the evolution equation for specific geometries, and could not compare their theoretical results with experimental measurements.

In the present work, the multiple-scale analysis of Ockendon & Ockendon is worked out for a two-dimensional rectangular basin of arbitrary depth. The periodic solution of the evolution equation is then obtained analytically. The results for the steady (harmonic) response amplitude compare remarkably well with the experimental data of Skalak & Yarymovych (1962) and is superior to their perturbation results. The internal resonance between the parametrically-excited dominant mode and a higher mode is also discussed.

Evolution Equation

We consider the one-half subharmonic resonant motion of an ideal, incompressible fluid in a two-dimensional rectangular tank forced to oscillate vertically. The length scales are normalized by half-width of the tank, and time is scaled by the frequency of the resonant waves $\omega=\omega_e/2$, where ω_e is the frequency of excitation. A coordinate system fixed with the tank is chosen with the origin and x-axis in the undisturbed free surface and z is positive upwards. The kinematic and dynamic boundary conditions on the free surface are

$$\frac{\partial\eta}{\partial t} + \epsilon\frac{\partial\eta}{\partial r}\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial z} = 0, \qquad \text{on } z=\epsilon\eta(x,t) \qquad (1a)$$

and

$$\frac{\partial\phi}{\partial t} + \frac{\epsilon}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + 4[N^2\mu + \epsilon\cos(2t)]\eta = 0, \qquad \text{on } z=\epsilon\eta(x,t) \qquad (1b)$$

where $\epsilon<<1$ is the nondimensional amplitude of the excitation, $\mu\equiv[\pi\tanh(\pi h)]^{-1}$, $N\equiv\Omega/\omega_e$; and $\Omega=[(\pi g/L)\tanh(\pi h)]^{1/2}$ is the dimensional linear natural frequency of the first (symmetric) standing wave in the tank. As a measure of tuning, we write $N=1/2+\lambda\epsilon$, where $\lambda=O(1)$ is the detuning parameter.

For the multiple-scale analysis, the slow-time scale $\tau=\epsilon t$ is introduced. The free-surface boundary conditions (1) are expanded in Taylor series about $z=0$, and ϕ and η are written in the perturbation series:

$$\phi = \epsilon^{-1/2}\phi_o + \phi_1 + \epsilon^{1/2}\phi_2 + \cdots, \qquad (2a)$$

$$\eta = \epsilon^{-1/2}\eta_o + \eta_1 + \epsilon^{1/2}\eta_2 + \cdots. \qquad (2b)$$

We proceed, in a manner similar to Ockendon & Ockendon, through $O(\epsilon^{1/2})$. Suppressing secularity at this third order, we obtain the evolution equation governing the complex amplitude $A(\tau)$ of the resonant response:

$$\mu \frac{dA}{d\tau} + 2i\mu\lambda A - iA^{*} - i\Gamma A|A|^{2} = 0 , \qquad (3)$$

where $\Gamma \equiv (6-5\mu^{2}\pi^{2}+16\mu^{4}\pi^{4}-9\mu^{6}\pi^{6})/(64\mu)$. If a weak, linear damping, α, is present in the dynamic system, the evolution equation (3) becomes simply:

$$\mu \frac{dA}{d\tau} + \alpha A + 2i\mu\lambda A - iA^{*} - i\Gamma A|A|^{2} = 0 , \qquad (4)$$

where $\alpha=\delta/\epsilon$, and δ is the ratio of the actual to the critical damping of the free oscillation of the resonant mode.

The phase-plane analysis of (3) & (4) has been considered by Miles (1984). The phase-plane trajectories can be classified into three different types for $\mu\lambda<0.5$, $\mu\lambda>0.5$ and $|\mu\lambda|<0.5$. For example, for $h=\infty$ and $\mu\lambda=1$, the undamped and damped ($\alpha=0.5$) phase-plane solutions for $A(\tau)\equiv C(\tau)+iD(\tau)$, where C,D are real, are shown in Figures 1 (a) and (b) respectively.

Solution of the Evolution Equation

Periodic solutions of the undamped evolution equation (3) can be obtained analytically. Representing the complex amplitude $A(\tau)$ as $A(\tau)\equiv a(\cos\gamma+i\sin\gamma)$, where a and γ are real functions of τ, (3) can be rewritten as:

$$\mu \frac{d\gamma}{d\tau} = - 2\mu\lambda + \Gamma a^{2} + \cos 2\gamma , \qquad (5a)$$

$$\mu \frac{da}{d\tau} = a \sin 2\gamma . \qquad (5b)$$

Eliminating τ from (5) and upon integration we obtain :

$$2a^{2}(\cos 2\gamma - 2\mu\lambda) + \Gamma a^{4} = E , \qquad (6)$$

where E is an integration constant. Equations (5b) and (6) can be further combined to give:

$$\frac{\mu}{a} \frac{da}{d\tau} = \pm \{ 1 - [2\mu\lambda+(E-\Gamma a^{4})/(2a^{2})]^{2} \}^{1/2} , \qquad (7)$$

which yields:

$$\tau = \pm \frac{\mu}{2} \int \frac{da^{2}}{[a^{4} - (E/2+2\mu\lambda a^{2}-\Gamma a^{4}/2)^{2}]^{1/2}} . \qquad (8)$$

Thus the slow time τ is expressed as an elliptic integral of the square of the amplitude a. At any specified τ, a^2 is given in terms of an elliptic function of τ, and the phase angle γ can be obtained from (6).

The period of the modulation, T, can be expressed explicitly:

$$T = g\mu K(k^{2})/\Gamma, \qquad (9)$$

where $K(k^2)$ is the complete elliptic integral of the first kind, and k, g are constants which depend on the specific phase-plane trajectories.

Steady Response

For steady (harmonic) response, the response amplitude z_o is defined as the maximum vertical distance between the trough and the crest of the free surface, so that z_o is given by twice the amplitude of A at the critical points which can be calculated directly from (3):

$$z_o = 2\sqrt{(2\mu\lambda \pm 1)/\Gamma}\ , \qquad \text{for } \mu\lambda > \begin{matrix}-1/2 \\ 1/2\end{matrix} \quad \begin{matrix}\text{stable response} \\ \text{unstable response.}\end{matrix}$$

For deep water, $\Gamma=\pi/8$, and the present preditions for z_o are compared to the third-order perturbation results and experimental data of Skalak & Yarymovych in Figure 2, where $\sigma=\omega/\Omega$. The comparison between our results and measurements is remarkably good and is better than that of the perturbation theory.

Internal Resonance

In perturbation analysis of weakly-nonlinear standing waves, an unique solution does not exist at some critical values of the fluid depth for which the natural frequency of a higher mode is an integral multiple of the fundamental frequency. Near these critical depths, internal resonance occurs between the parametrically-excited dominant mode and a higher mode. For a circular basin, Miles (1984) analysed the two-to-one (i.e., $\Omega_2=2\Omega_1$) internal resonance between the parametrically-resonant first antisymmetric mode and the first axisymmetric mode.

Unlike the resonant waves in a circular tank, the first possible two-to-one internal resonance in a two-dimensional rectangular basin is between the first symmetric and the third symmetric or antisymmetric modes. We define the detuning parameter for the q^{th} internally-resonant mode:

$$\frac{\Omega_q}{\omega_e} = 1 + \lambda_q\epsilon\ , \tag{10}$$

where Ω_q is the dimensional linear natural frequency of the q^{th} mode, and consider, for simplicity, only perfectly-tuned resonance. Following a similar procedure as before, we obtain the equations governing the evolution of the complex amplitudes of the dominant (first) mode $A(\tau)$ and that of the q^{th} mode $A_q(\tau)$:

$$\mu\frac{dA}{d\tau} + 2i\mu\lambda A - iA^* - i\Gamma A^2A^* - iAA_q^2\Sigma = 0\ , \tag{11a}$$

and

$$2\mu_q\frac{dA_q}{d\tau} + 8i\mu_q\lambda_qA_q - i\Gamma_qA_q^2A_q^* - iA_qA^2\Sigma_q = 0\ , \tag{11b}$$

where $\mu_q=[q\pi\tanh(q\pi h)]^{-1}$ and $\Gamma,\Sigma,\Gamma_q,\Sigma_q$ are functions of q and h. Thus, for the two-dimensional rectangular tank, the nonlinear coupling interaction between two internal resonant modes are cubic, rather than quadratic as in the case of the circular basin analysed by Miles.

References

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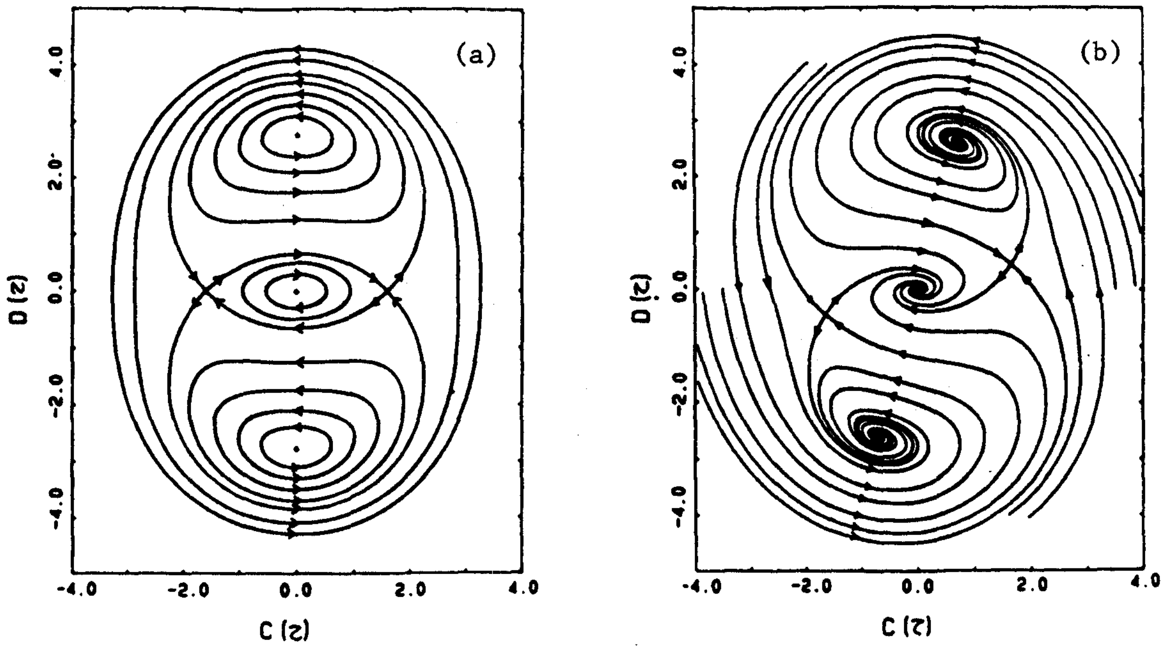

Figure 1. Phase-plane solution of the (a) undamped, and (b) damped (α=.5) case for h=∞ and $\mu\lambda$=1.

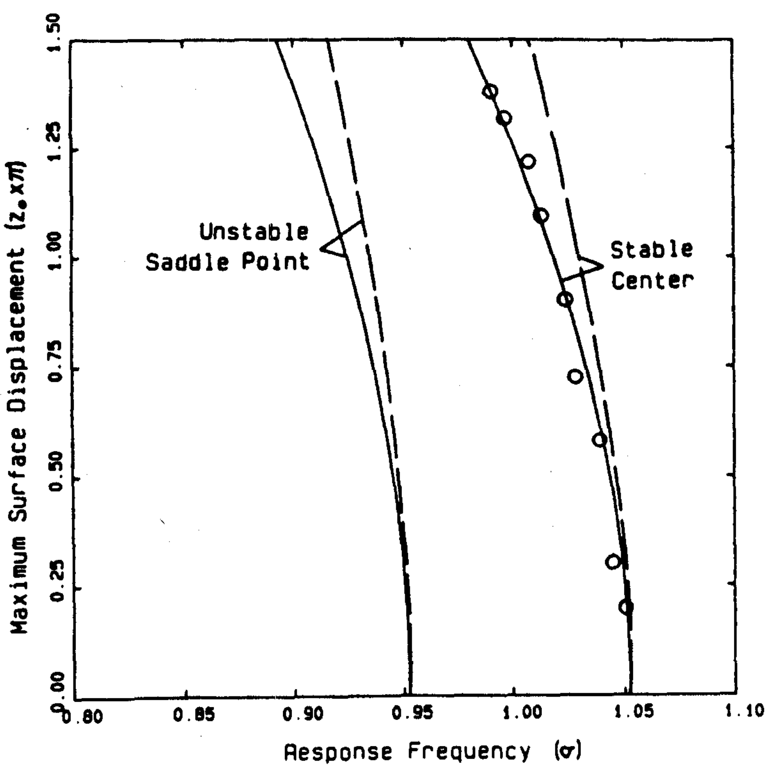

Figure 2. Comparisons between the present theory (————), and the third-order perturbation results (— — —) and experimental measurements (o) of Skalak & Yarymovych (1962), for the harmonic response as a function of the response frequency $\sigma \equiv \omega/\Omega$.

Evans: What connection does your work have with the recent work of Bridges in *JFM* on sloshing in rectangular tanks?

Tsai & Yue: The evolution equations in the cross-waves problem for the internal resonance between longitudinal forced waves and transverse cross-waves are similar to the equations in Bridges' paper. However we have a forcing term in the equation of longitudinal waves and we are not sure if it has the same property as Bridges' problem. For the internal resonance in a vertically heaving tank, the evolution equation has an A^* term which comes from the parametric resonance. We are presently studying the dynamic properties of these evolution equations.