STABILITY ANALYSIS OF FREE-SURFACE PANEL METHODS FOR THE WAVE RESISTANCE PROBLEM

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The numerical solution of linearized formulations of the wave resistance problem by a distribution of panels over the ship hull and the free surface, using the Rankine source $1/r$ as the Green function, has in recent years enjoyed increased popularity. The fundamental numerical properties of the free-surface discretization are systematically studied in three dimensions. In particular the a) consistency, b) order, c) numerical dissipation and dispersion, d) stability, e) enforcement of the radiation condition and f) linear-system diagonal dominance are considered for high- and low-order free-surface discretizations, in connection with the source distribution method as well as its transpose formulation obtained from the direct application of Green's theorem.

As is customary in analogous studies in computational fluid dynamics, the numerical analysis is considered of a simple relevant flow which often accepts a closed-form solution. Here we select the wave flow generated by a submerged Havelock source. An integral equation is first derived in the continuous problem for the source strength or velocity potential distributions on the free surface. Its numerical approximation is then obtained by discretizing a finite part of the free surface with rectangular panels, combined with a polynomial variation for the unknown function over each panel, a finite-difference scheme to approximate the convective terms in the free-surface condition and an appropriate enforcement of the radiation condition. The analysis of the numerical properties of such free-surface discretizations is illustrated for a low-order scheme and in connection with the integral equation for the velocity potential. For the two-dimensional submerged source, a study of these properties has been carried out by Piers (1983).

Consider the three-dimensional potential flow due a point Rankine source submerged at a distance d below the free surface, translating in the positive x-direction with velocity U. The velocity potential is subject to the Neumann-Kelvin condition on the mean position of the free surface, and via an application of Green's theorem it can be shown to satisfy the following integral equation on the $z = 0$ plane

$$2\pi\varphi(x,y) + \frac{U^2}{g}\iint_{z=0} d\xi d\eta \frac{\varphi_{\xi\xi}(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]} = \frac{1}{(x^2+y^2+d^2)^{1/2}} \equiv F(x,y). \qquad (1)$$

This integral equation is of convolution type and accepts a closed form solution which is the Havelock wave source potential. Taking the double (u,v) Fourier transform of (1) with respect to the (x,y) coordinates, it follows that

$$\tilde{\varphi}(u,v) = \frac{\nu}{2\pi}\frac{\tilde{F}(u,v)}{\tilde{W}(u,v)} \qquad (2)$$

$$\tilde{W}(u,v) = \nu + i\epsilon - \frac{u^2}{(u^2+v^2)^{1/2}}, \qquad (3)$$

where $\nu = g/U^2$ and ϵ is a small positive constant, known as the Rayleigh visocity, which ensures that the waves associated with the disturbance obtained by the Fourier inversion of (2) appear downstream. This elegant device is particularly effective with Fourier analysis. In what follows, it will be shown that one way of introducing the proper "Rayleigh viscosity" in the numerical solution of (1) is to select an upstream difference scheme for the approximation of the second ξ−derivative of the velocity potential φ.

Consider the numerical solution of equation (1) by discretizing the $z = 0$ plane and the unknown velocity potential, using:

- Rectangular panels with sides parallel to the x− and y−axes of lengths h_x and h_y respectively
- A piecewise constant variation of φ and F over the surface of each panel
- The three-node upstream difference formula relating φ_{xx} to the local and two upstream values of φ.
- The panel centroids as the collocation points.

Employing this discretization in (1), assuming that the extent of the free-surface discretization is infinite and taking its discrete Fourier transform, it follows that

$$\hat{\varphi}(u,v) = \frac{\nu}{2\pi} \frac{\hat{F}(u,v;h_x,h_y)}{\hat{W}(u,v;h_x,h_y)} \tag{4}$$

$$\hat{W}(u,v;h_x,h_y) = \nu - \frac{u^2}{(u^2+v^2)^{1/2}} e^{-ih_x u} \left(\frac{\sin \frac{h_x u}{2}}{\frac{h_x u}{2}} \right)^3 \frac{\sin \frac{h_y v}{2}}{\frac{h_y u}{2}}, \tag{5}$$

where $\hat{\varphi}$ is the discrete Fourier transform of the solution velocity potential. The "proximity" of the discrete to the continuous solution hinges upon the proximity of $\hat{F}$ to $\tilde{F}$ and of $\hat{W}$ to $\tilde{W}$. More intimate, however, is the relation between the functions $\tilde{W}, \hat{W}$, since it is the location of the root(s) of $\hat{W}$ that is primarily responsible for the waves represented by the numerical solution. Most items in the ensuing discussion, except for the radiation condition and the linear system diagonal dominance, are concerned with *interior* properties of the free-surface discretization therefore justifying the assumption that the extent of the free-surface grid is infinite.

We are now in the position to study the basic properties of the adopted discretization:

a) *Consistency*: The discretization of the continuous problem is consistent if the functions $\hat{F}, \hat{W}$ tend to their continuous counterparts $\tilde{F}, \tilde{W}$ in the limit as h_x, h_y tend to zero, uniformly in u, v. For the model numerical scheme considered here, it is easily confirmed that $\hat{W} \to \tilde{W}$ by comparing equations (3) and (5). This can be also shown to hold for the function F.

b) *Order*: The order of the discretization is determined by the slowest of the two rates at which $\hat{F}, \hat{W}$ approach $\tilde{F}, \tilde{W}$ respectively. Expressing the order of their differences in terms of h_x^m, h_y^n in the limit as $h_x, h_y \to 0$ and uniformly in u, v, the minimum of the two integers m, n defines the order of the discretization. To ensure consistency, it is necessary that this minimum integer is positive.

c) *Numerical damping and dispersion*: The order of a given discretization is a global property which, unlike the informal presentation in b), is rigorously defined as the integrated difference in some norm between the numerical and continuous solutions. Of particular interest in the present wave flow is the pointwise comparison of the numerical and continuous wave disturbances. Two characteristic measures of their difference are the *damping* and *dispersion* introduced by the discretization. They can be quantified by comparing the relative location of the roots of $\tilde{W}$ and $\hat{W}$. Without loss of generality in the illustration of the principal idea, we set $v = 0$ in (3) and (5) or else we consider the corresponding two-dimensional flow. The root of $\tilde{W}$ occurs at $|u| = \nu$, while the corresponding *principal* root of $\hat{W}$ can be obtained from (5) for small h_x in the form

$$\hat{\nu} = \nu(1 + ih_x - \frac{11}{8}h_x^2) + O(h_x^3). \tag{6}$$

Its displacement in the positive imaginary half-plane by a distance $i\nu h_x$ indicates that the numerical solution introduced numerical damping of $O(h_x)$. Its displacement in the direction of the real axis by $-\frac{11}{8}\nu h_x^2$ indicates that the corresponding numerical dispersion is of $O(h_x^2)$. In practice these orders tax so severly the smallness of h_x, or else the maximum number of panels necessary for acceptable accuracy, that the derivation of a higher-order scheme is necessary.

d) *Numerical stability*: The principal root of $\hat{W}$ is responsible for the generation of the physical wave disturbance. Other *spurious* roots, real or complex, are the creation of the numerical approximation and their location determines its stability properties. Real spurious roots will contaminate the physical wave distrurbance generated by the principal root with unphysical oscillations. On the one-dimensional grid with panel size h_x ($v = 0$ case), the most rapidly oscillatory disturbance has a minimum wavelength $2h_x$, or a maximum wavenumber $u^* = \pi/h_x$. Spurious roots with small or zero imaginary part and modulus less than u^*, will generate a visible oscillatory error. If their modulus is greater than u^*, they still generate an oscillatory error, which is however not visible as a disturbance with sign alternating at the expected frequency on our grid, since its wavelength is smaller than $2h_x$. This error is nevertheless present, its amplitude is proportional to the modulus of the residue of the complex function (4) at the spurious root and its manifestation is known under the name *aliasing*. Spurious roots with substantial imaginary parts are harmless, since the oscillatory error they create is exponentially small. In summary, a given discretization will be called unstable when spurious roots are located near the real u−axis, with more severe errors usually occuring when their modulus is less than u^*.

e) *Radiation condition*: The principal root of $\hat{W}$ defined by (6) is shifted in the positive imaginary u−plane. Recalling the discussion on the continuous problem following equation (3), we may conclude that this ensures that the disturbance generated by this root represents no waves upstream, thus satisfying the radiation condition. Numerical waves may still be propagating upstream if a spurious root is real. Therefore, the radiation condition is strictly enforced when the imaginary part of all spurious roots remains finite in the limit $h_x \to 0$. The proper shift of the principal root in the complex plane is ensured by the sign of the exponent in $e^{-ih_x u}$ in (5) which is generated by the Fourier transform of the upstream difference operator. Had a downstream difference operator been selected, this factor would be equal to $e^{ih_x u}$, and by virtue of (6) would cause the waves associated with the principal disturbance to propagate upstream. For a symmetric difference operator, the same factor takes the value $\cos(h_x u)$ which produces a real $\hat{W}$, therefore not permitting the enforcement of the radiation condition via the finite-difference scheme. On a free-surface grid of finite extent, the radiation condition may however be enforced by proper conditions imposed at its boundaries. Numerical examples will be presented which demonstrate the feasibility and performance of this alternative approach. An advantage of centered-difference schemes is that they are responsible for a smaller numerical damping than upstream difference schemes of the same order. This can be seen by substituting the Taylor series expansions of $e^{-ih_x u}$ and $\cos(h_x u)$ for small h_x in (5) and evaluating their effect in the complex displacement of the root $\hat{\nu}$ from its continuous value ν.

f) *Linear System Diagonal Dominance*: In practice it is very desirable to be able to develop an iterative scheme for the solution of the linear system obtained from the discretization of equation (1). The numerical treatment of wave flows around realistic ship forms typically requires the use of large numbers of panels over the ship hull and free surface, especially at low Froude numbers. Therefore, iterative solutions lead to substantial computational gains relative to direct Gauss reduction. Our understanding of the spectral properties of the integro-differential equation (1) enforced over a finite domain of the free surface is limited. We believe that the finite extent of the free-surface grid, the conditions imposed at its boundary and the type of iterative scheme used are critical factors in the determination of its rate of convergence. In spite of the pessimistic picture painted by Forbes (1984) concerning the convergence of iterative schemes for equations of this type, an accelerated Gauss-Sidel scheme was found to converge for low and moderate, but not for high Froude numbers. Research on this topic is ongoing.

Numerical experiments demonstarting the above numerical analysis will be presented for the solution of the flow generated by the Havelock source. Attention will be finally placed on the design and performance of a high-order scheme using a quadratic-spline approximation of the velocity potential on the free surface in both the $x-$ and $y-$directions, which was found to be both accurate and economical for the numerical solution of elementary wave disturbances as well as realistic flows past ship hulls.

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