OSCILLATING IMMERSED PLATES AND HYPERSINGULAR INTEGRAL EQUATIONS

by

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Dock problems are classical, but still attract attention. Thus, Miles (1987) has recently considered the forced harmonic oscillations of a rigid circular plate in the free surface of deep water. Numerically, such problems are relatively straightforward, for they can always be reduced to a Fredholm integral equation of the second kind over the wetted surface of the dock, D (see, e.g. MacCamy, 1961):

$$\varphi(p) + K \int_D \varphi(q) G(p,q) ds_q = \int_D V(q) G(p,q) ds_q .$$

Here, p and q are points on D, φ is the velocity potential, $\partial\varphi/\partial n = V$ on D, $K = \omega^2/g$, ω is the radian frequency, and G is the usual fundamental solution.

The situation is quite different if the dock is submerged. Thus, for an oscillating immersed thin plate, the potential must have a representation as

$$\varphi(P) = \int_D [\varphi(q)] \frac{\partial}{\partial n_q} G(P,q) ds_q \tag{1}$$

for P in the water, i.e. as a distribution of normal dipoles over (one side of) the thin plate, with density equal to $[\varphi]$, the discontinuity in φ across the plate. (Here, we have assumed that the plate is infinitesimally thin. However, the representation (1) is appropriate for floating bodies with a finitely-thin component.) Application of the boundary condition gives

$$\frac{\partial}{\partial n_p} \int_D [\varphi(q)] \frac{\partial}{\partial n_q} G(p,q) ds_q = V(p), \quad p \text{ on } D, \tag{2}$$

which is an equation for $[\varphi]$.

One would like to take the operator $\partial/\partial n_p$ under the integral in (2), but this leads to a non-integrable kernel. To avoid this, many authors have advocated various regularizations. Alternatively, it can be proved that (2) can be rewritten as

$$\oint_D [\varphi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p,q) ds_q = V(p), \quad p \text{ on } D, \qquad (3)$$

where the integral has to be interpreted as a Hadamard finite-part integral. Equation (3) is a hypersingular integral equation.

As an example, consider a submerged smooth plate in two dimensions. Then, (3) can be written as

$$\oint_{-1}^{1} [\varphi(t)] \left\{\frac{1}{(x-t)^2} + K(x,t)\right\}dt = v(x), \quad -1 < x < 1,$$

where $v(x)$ is known, $K(x,t)$ is a weakly-singular kernel and $[\varphi(t)]$ is to be found. The finite-part integral is defined by

$$\oint_{-1}^{1} \frac{f(t)}{(x-t)^2}\, dt = \lim_{\varepsilon\to 0} \left\{ \int_{-1}^{x-\varepsilon} \frac{f(t)}{(x-t)^2}\, dt + \int_{x+\varepsilon}^{1} \frac{f(t)}{(x-t)^2}\, dt - 2\,\frac{f(x)}{\varepsilon}\right\}$$

which compares with

$$⨍_{-1}^{1} \frac{f(t)}{x-t}\, dt = \lim_{\varepsilon\to 0} \left\{ \int_{-1}^{x-\varepsilon} \frac{f(t)}{x-t}\, dt + \int_{x+\varepsilon}^{1} \frac{f(t)}{x-t}\, dt\right\}$$

for the Cauchy principal-value integral of f.

Computational experience with hypersingular integral equations is scarce and

and scattered, but accumulating. Various methods for their solution will be described and compared.

References

MacCamy, R.C. On the scattering of water waves by a circular disk. Arch. Rat. Mech. Anal. 8 (1961) 120-138.

Miles, J.W. On surface-wave forcing by a circular disk. J. Fluid Mech. 175 (1987) 97-108.

Newman: The questions which have been raised regarding the convergence of piecewise-constant unknowns and appropriate numerical algorithms are largely answered in the aerodynamics literature. Cosine-spacing is very useful, with the collocation points at mid-points in the "angular" coordinate. Extensive discussion can be found in a paper by Lan (*J. Aircraft,* 1974).

Martin: It is well known that cosine spacing is appropriate for integral equations with a Cauchy principal value (CPV) kernel. Since the finite-part integral (defined in the abstract) is $(-d/dx)$ of the CPV integral, it is not surprising that cosine spacing is still appropriate. Expansions in terms of Chebyshev polynomials can also be used (Kaya & Erdogan, *Quart. Appl. Math.*1987). However, the real problems are in three dimensions!

Xu: Your work on the integral equation of the 1st kind is interesting. As we know, this type of integral equation often occurs in the analysis of a lifting body. In the two-dimensional unsteady case, or the three-dimensional case, the integral domain always includes the wake region behind the body. How would you extend your theorems to account for these cases?

Martin: The formula

$$\frac{\partial}{\partial n_p}\int_\Gamma [\phi(q)]\frac{\partial}{\partial n_q}G(p,q)ds_q = \neq\!\!\!\!\int_\Gamma [\phi(q)]\frac{\partial^2}{\partial n_q \partial n_q}G(p,q)ds_q$$

is valid at any point p on Γ at which the jump $[\phi]$ is at least continuously differentiable ($[\phi]$ in $C^{1,\alpha}(\Gamma)$: $[\phi]$ has a tangential derivative which is itself Hölder continuous). In your problem, Γ is the lifting body plus wake. (If p is at either end of Γ, then the finite-part integral must be replaced by a different finite-part integral.) The formula above is also valid in 3-D. It is also assumed that Γ is smooth (twice differentiable).