MASS TRANSPORT IN WAVE FLUMES

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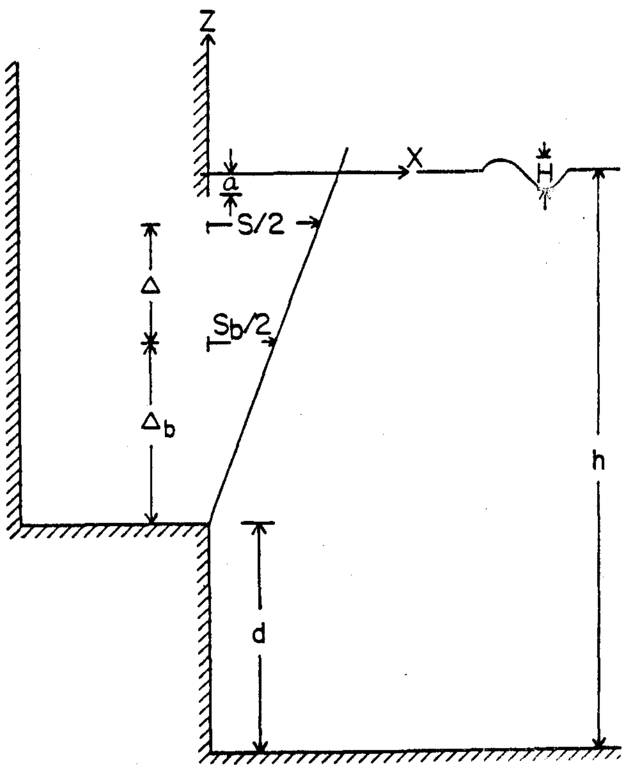
ABSTRACT. Previous expansions of the perturbation solution to the weakly nonlinear boundary value problem for a hinged wavemaker of variable-draft have not included the time-independent contributions in a completely rigorous manner. Time-independent forcing data appear in the Neumann boundary conditions at second-order on both the wavemaker and free surface boundaries. Orthonormal eigenseries may be used to satisfy these time-independent boundary conditions. The horizontal derivative of the leading order term in this eigenseries is shown to be exactly equal in magnitude, but opposite in direction, to the Eulerian Stokes drift. Previous estimates of this mean, uniform over depth, return current have been computed from a conservation of mass flux principle. Profiles of the Lagrangian-induced streaming currents computed from the second-order time-independent eigenseries illustrate the effects of the irregular points on the accuracy of the second-order solution.

NONLINEAR WAVEMAKER THEORY

All physical variables (denoted by superscript asterisks, *) will be made dimensionless by the following: $(x,z,h,d,b,\Delta,L) = k^*(x^*,z^*,h^*,d^*,b^*,\Delta^*,L^*)$; $(t,T) = \sqrt{g^*k^*}\,(t^*,T^*)$; $(H,\eta,S,\xi,\chi) = (H^*,\eta^*,S^*,\xi^*,\chi^*)/a^*$; $(u,w) = (u^*,w^*)/(a^*\sqrt{g^*k^*})$; $\Phi = \Phi^*/(a^*\sqrt{g^*/k^*})$; $B = B^*/(a^*g^*)$; and $p = p^*/(\rho^*a^*g^*)$ where a^* = amplitude of the first-harmonic wave component; $k^*(= 2\pi/L^*)$ = the wave number; L^* = wave length; g^* = gravitational constant; ρ^* = fluid mass density; and T^* = wave period = the period of the wavemaker oscillation.



$$\chi(z) = \xi(z)\{U(z+h-d)-U(z+b)\}\sin \omega t$$

$$\xi(z) = (S/2)/(\Delta/h)[M(1+z/h)+B']$$

$$M = 1-S_b/S$$

$$B' = \Delta/h-M(d/h+\Delta_b/h+\Delta/h)$$

Fig. 1. Definition sketch for generic wavemaker.

A generic wavemaker is shown in Fig. 1 which generates two-dimensional, irrotational motion of an inviscid, incompressible fluid in a semi-infinite channel of constant, still water depth, h.

The solution to the second-order problem may be expressed as the linear sum of four scalar velocity potentials given by ${}_2\Phi = {}_2\Phi^s + {}_2\Phi^e + {}_2\Phi^f + \Psi$ in which ${}_2\Phi^s$ is a second-order Stokes wave potential; ${}_2\Phi^e$ is a near-field evanescent interaction potential; ${}_2\Phi^f$ is a wavemaker-forced potential; and Ψ is a time-independent potential needed to satisfy exactly the following two boundary conditions:

$$\mathcal{E}\{{}_2\Phi^s+{}_2\Phi^e+{}_2\Phi^f+\Psi\} = a_1^2 f_1(\phi_1)\sin 2(x-\tau) - a_1\sin(x-2\tau)\sum_{m=2} a_m\exp(-\alpha_m x) f_2(\phi_1,\phi_m)$$

$$+a_1\cos(x-2\tau)\sum_{m=2} a_m\exp(-\alpha_m x) f_3(\phi_1,\phi_m) - \sin 2\tau \sum_{m=2}\sum_{n=2} a_m a_n \exp[-(\alpha_m+\alpha_n)x] f_4(\phi_m,\phi_n)$$

$$-a_1\cos x\sum_{m=2} a_m\exp(-\alpha_m x) f_5(\phi_1,\phi_m) \;;\; x \geq 0,\; z = 0$$

$$\frac{\partial}{\partial x}\{{}_2\Phi^s+{}_2\Phi^e+{}_2\Phi^f+\Psi\} = \frac{a_1}{2}\, W_1(\phi_1,\xi,z)[1-\cos 2\tau] + \frac{\sin 2\tau}{2}\sum_{m=2} a_m\alpha_m W_2(\phi_m,\xi,z);\; x=0,\; -h\leq z\leq 0$$

in which the nonlinear, free surface interaction terms f_1, f_2, f_3, f_4, and f_5, and the nonlinear wavemaker interaction terms W_1 and W_2 represent nonlinear interactions involving first-order quantities; and $\mathcal{E}(\cdot) = (\omega_o^2\partial^2/\partial\tau^2 + \partial/\partial z)(\cdot)$.

The second-order, near-field evanescent potential, ${}_2\Phi^e$, is given by

$${}_2\Phi^e(x,z,\tau) = a_1\cos(x-2\tau)\sum_{m=2} a_m\exp(-\alpha_m x)[A_m\phi_1(z)\phi_m(z) + B_m\phi_1'(z)\phi_m'(z)]$$

$$-a_1\sin(x,2\tau)\sum_{m=2} a_m\exp(-\alpha_m x)[A_m\phi_1'(z)\phi_m'(z) - B_m\phi_1(z)\phi_m(z)]$$

$$-\sin 2\tau\sum_{m=2}\sum_{n=2} a_m a_n\exp[-(\alpha_m+\alpha_n)x]C_{mn}[\phi_m(z)\phi_n(z) - \phi_m'(z)\phi_n'(z)]$$

$$A_m = \frac{\alpha_m^2}{\omega_o}\,\frac{[3(4\omega_o^4+\alpha_m^2-1)+2\omega_o^4]}{[(4\omega_o^4+\alpha_m^2-1)^2+4\alpha_m^2]} \;;\; B_m = \frac{-\alpha_m}{2\omega_o}\,\frac{[(4\omega_o^4+\alpha_m^2-1)^2+2\omega_o^4(4\omega_o^4+\alpha_m^2-1)-8\alpha_m^2]}{[(4\omega_o^4+\alpha_m^2-1)^2+4\alpha_m^2]}$$

$$C_{mn} = \frac{\alpha_m\alpha_n}{4\omega_o}\,\frac{[(\alpha_m+\alpha_n)^2 + 2\alpha_m\alpha_n + 6\omega_o^4]}{[(\alpha_m-\alpha_n)^2 + 4\omega_o^4]}$$

The second-order, wavemaker-forced potential, ${}_2\Phi^f$, is given by

$${}_2\Phi^f(x,z,\tau) = \{E_1\cos(\beta_1 x-2\tau)+F_1\sin(\beta_1 x-2\tau\}Q_1(z) - \sum_{j=2}\exp(-\beta_j x)\{E_j\sin 2\tau + F_j\cos 2\tau\}Q_j(z)$$

$$E_j = -\beta_j^{-1}\{a_1\sum_{m=2} a_m[(A_m+\alpha_m B_m - \frac{\alpha_m^2}{2\omega_o})\langle\phi_1\phi_m,Q_j\rangle_z + (B_m-\alpha_m A_m - \frac{\alpha_m}{2\omega_o})\langle\phi_1'\phi_m',Q_j\rangle_z]$$

$$+\sum_{m=2}\sum_{n=2} a_m[(\alpha_m+\alpha_n)[C_{mn}+\frac{\alpha_m\alpha_n}{4\omega_o}][\langle\phi_m\phi_n,Q_j\rangle_z - \langle\phi_m'\phi_n',Q_j\rangle_z]\} \;;\; j\geq 1$$

$$F_j = \beta_j^{-1}a_1^2\{(\frac{3-5\omega_o^4}{4\omega_o^5})[\langle\phi_1^2,Q_j\rangle_z + \langle\phi_1'^2,Q_j\rangle_z] + \sum_{m=2}(a_m/a_1)[(A_m\alpha_m - B_m + \frac{\alpha_m}{2\omega_o})\langle\phi_1\phi_m,Q_j\rangle_z$$

$$+(A_m+\alpha_m B_m - \frac{\alpha_m^2}{2\omega_o})\langle\phi_1'\phi_m',Q_j\rangle_z]\} \;;\; j\geq 1$$

The second-order, time-independent, free surface potential, Ψ^{fs}, is given by

$$\Psi^{fs}(x,z) = a_1 \cos x \sum_{m=2} a_m \exp(-\alpha_m x)[b_m \phi_1(z)\phi_m(z) + c_m \phi_1'(z)\phi_m'(z)]$$
$$- a_1 \sin x \sum_{m=2} a_m \exp(-\alpha_m x)[b_m \phi_1'(z)\phi_m'(z) - c_m \phi_1(z)\phi_m(z)]$$

$$b_m = -\frac{\alpha_m^2}{\omega_o(\alpha_m^2+1)} \quad ; \quad c_m = -\frac{\alpha_m}{2\omega_o}\frac{(\alpha_m^2-1)}{(\alpha_m^2+1)}$$

The second-order, time-independent, wavemaker potential, Ψ^{wm}, is given by

$$\Psi^{wm}(x,z) = \sum_{j=0} d_j \psi_j(z)[\exp(-\mu_j x) + \delta_{jo}(x-1)]$$

$$\psi_j(z) = \cos\mu_j(z+h)/[h/(2-\delta_{jo})]^{1/2} \quad ; \quad \mu_j = j\pi/h \; ; \; j \geq 0$$

$$d_j = (2\psi_j(o)/\omega_o)[\delta_{jo}-\mu_j]^{-1}\{[4+\mu_j^2]^{-1} - \omega_o\mu_j^2 \sum_{m=2} a_m \alpha_m \phi_m(o)[(1+\mu_j^2-\alpha_m^2)^2+4\alpha_m^2]^{-1}\} \quad ; \; j \geq 1$$

MEAN HORIZONTAL MOMENTUM

The time- and depth-averaged horizontal momentum per unit area is

$$M_{E(L)} = \langle\int_{-h}^{\eta} U_{E(L)} dz\rangle_{2\pi}$$

where $\langle\cdot\rangle_{2\pi} = (2\pi)^{-1}\int_o^{2\pi}(\cdot)d\tau$ and $U_{E(L)}$ is an Eulerian (Lagrangian) velocity.

Eulerian. The Eulerian velocity is given by $M_E = \bar{U}_\Psi + \bar{U}_\Phi$ where

$$\bar{U}_\Psi = -\epsilon\int_{-h}^{o}\frac{\partial\Psi}{\partial x}dz = U_{\Psi,\infty}(d_o) + U_{\Psi,e}(\alpha_m h, x)$$

$$\bar{U}_\Phi = -\epsilon\omega_o\langle(\partial_1\Phi/\partial x)(\partial_1\Phi/\partial\tau)\rangle_{2\pi} = U_{\Phi,\infty}(\omega_o) + U_{\Phi,e}(\alpha_m h, x) \quad ; \; z = 0$$

The dimensionless far-field component, $U_{\Psi,\infty}(d_o)$, is given by

$$U_{\Psi,\infty}(d_o) = -\epsilon d_o\sqrt{h} = -\epsilon(2\omega_o)^{-1}$$

which is exactly equal to the mean return current in a closed wave flume! This quantity is usually estimated from a conservation of mass flux principle. The dimensionless evanescent component, $U_{\Psi,e}(\alpha_m h,x)$, is given by

$$U_{\Psi,e}(\alpha_m h,x) = \epsilon(2\omega_o^2)^{-1}\cos x \sum_{m=2} a_m\phi_m(o)[\alpha_m - \tan x]\exp{-\alpha_m x}$$

The dimensionless far-field component, $U_{\Phi,\infty}(\omega_o)$, is given by

$$U_{\Phi,\infty}(\omega_o) = \epsilon(2\omega_o)^{-1}$$

which is the Eulerian Stokes drift that is exactly canceled by $U_{\Phi,\infty}(d_o)$! The dimensionless evanescent component, $U_{\Phi,e}(\alpha_m h,x)$, is given by

$$U_{\Phi,e}(\alpha_m h,x) = (\epsilon/2)\cos x \sum_{m=2} a_m\phi_m(o)[\alpha_m - \tan x]\exp{-\alpha_m x}$$

Lagrangian. The Lagrangian velocity may be estimated from the Eulerian velocity by, approximately

$$\vec{u}_L = -\epsilon\vec{\nabla}\Psi + (\epsilon/\omega_o)\langle(\int^{\tau} \vec{\nabla}_1\Phi d\tau')\cdot\vec{\nabla}(\vec{\nabla}_1\Phi)\rangle_{2\pi} + O(\epsilon^2)$$

where the Lagrangian velocity $\vec{u}_L = [u_L, v_L]$. The horizontal component $u_L(x,z)$, is, approximately

$$u_L(x,z) = -\epsilon\,\frac{\partial\Psi}{\partial x} + \frac{\epsilon}{\omega_o}\,\{\frac{\cosh 2(h+z)}{\sinh 2h} + (a_1/2)\cos x$$

$$\sum_{m=2} a_m\alpha_m \exp{-\alpha_m x}[\phi_1'(z)\phi_m'(z)(\alpha_m - \tan x) - \phi_1(z)\phi_m(z)(\alpha_m \tan x + 1)]\}$$

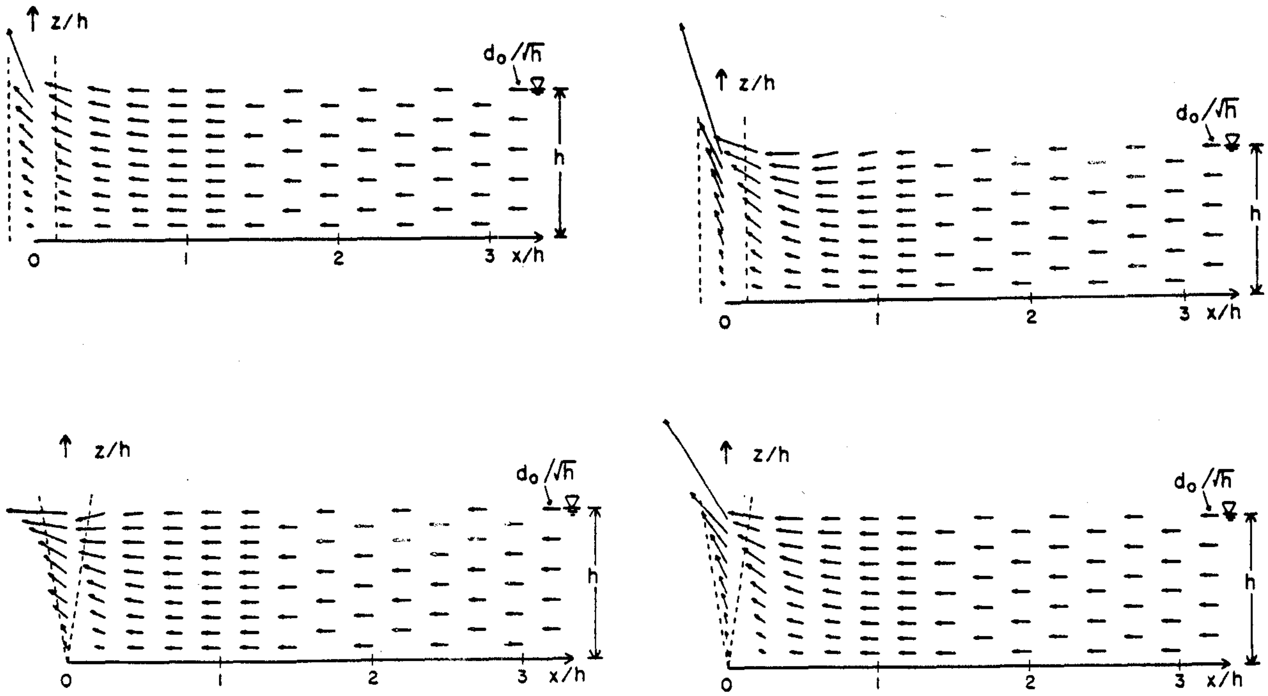
and the vertical component, $v_L(x,z)$, is, approximately

$$v_L(x,z) = -\epsilon\,\frac{\partial\Psi}{\partial z} - \epsilon(2\omega_o)^{-1}a_1\cos x \sum_{m=2} a_m\alpha_m \exp{-\alpha_m x}[\phi_1(z)\phi_m'(z)(\alpha_m \tan x - 1)$$

$$+ \phi_1'(z)\,\phi_m(z)(\tan x + \alpha_m)]$$

The magnitude of the time independent velocity $|\vec{u}_L|$ is illustrated below.

$h/L_o = 0.2$ $h/L_o = 0.5$



Ursell: The terms in your potential ${}_2\Phi^e$ should be harmonic functions but it is not obvious that they are. A detailed explanation would be appreciated.

Hudspeth: The ${}_2\Phi^e$ potential is required to satisfy the inhomogeneous free-surface forcing which is composed of products of elementary transcendental functions from the first-order solution. Although it is not obvious that a careful combination of these products of elementary transcendental functions will be a harmonic function, a typical term in the double summation series can be seen to be given by

$$\varphi_{mn} = C_{mn} exp[-(\alpha_m + \alpha_n)x]\,[\varphi_m(z)\varphi_n(z) - \varphi'_m(z)\varphi'_n(z)]$$
$$= \frac{C_{mn}}{n_m n_n} exp[-(\alpha_m + \alpha_n)x] cos(\alpha_m + \alpha_n)(z+h)$$

$$\varphi_{mn} = \frac{C_{mn}}{n_m n_n} exp[-K_{mn}x] cos K_{mn}(z+h)$$

which is a harmonic function. A typical term in each of the two single summation series may then be obtained from a typical φ_{mn} term, by letting the separation constant be complex-valued ($\alpha_n = +i$, say) and $C_{mn} = A_m - iB_n$. Each term in the single summation series becomes a product of elementary transcendental functions which is a harmonic function having a complex-valued separation constant.