

## Iterative Solutions of Floating Body Integral Equations

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Various integral equation formulations have been given for the floating body problem with linearized free surface condition. Best known is the famous equation derived by John [1] which is an equation of the second kind over the wetted portion of the body. This equation is not uniquely solvable at the so called irregular frequencies. Even away from irregular frequencies the integral operator is not compact in the classical function spaces of either continuous or square integrable functions on the boundary, unless the body intersects the free surface at right angles. This restricts the applicability of the Fredholm alternative to this case. The equation remains valid for non-normal intersection though uniqueness implies existence only with the help of potential theory for irregular domains as discussed, e.g. in [2] and [3]. The irregular frequency problem may also be resolved by modifying the integral equation as reported in [4] and [5], but as before the integral operator is not compact.

Alternatively an integral equation free of irregular frequencies may be derived using a much simpler Green function, one that satisfies the boundary condition on the bottom of the fluid in the finite depth case but which does not satisfy the free surface condition. This again takes the form of a second kind integral equation, this time over both the submerged portion of the body and the entire free surface. Again the operator is not compact, in classical spaces, so uniqueness and existence must be established independently. Uniqueness was treated in [6] while existence has not yet been proven.

Despite the complications of non-compactness and irregular frequencies these various integral equations have served as the basis for numerical treatment using collocation or panel methods. Here the integral equation is discretized yielding a system of algebraic equations which is solved, usually by Gaussian elimination.

There exist a variety of iterative methods of solving matrix equations, the simplest of which, perhaps, is the relaxation or one step Richardson method, e.g. [7]. These involve transforming the original equation through the introduction of an acceleration parameter the choice of which depends on the spectrum of the matrix. This method has had an analogous development in operator equations but has not been widely applied, if at all, to many of the linear integral equations of mathematical physics. Recently it has been shown how this iterative method may be applied to integral equations of acoustic scattering [8]. In the present paper it is shown how the method may be applied to the integral equation formulations of the floating body.

### 2. Notation and Formulation of the Problem

For definiteness, we treat here the three-dimensional floating body problem with finite depth  $h$ . As usual, we denote the fluid domain by  $D_+$ , the hull by  $C_0$ , the free surface by  $C_f$  and the bottom by  $C_B$ . Furthermore, we denote by  $C_w$  the projection of the ship hull on the free surface. One may formulate the floating body problem in terms of the potential function  $\phi$  satisfying

$$\begin{aligned}\nabla^2 \phi &= 0 \text{ in } D_+, \quad \frac{\partial \phi}{\partial n} = V \text{ on } C_0, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } C_B \\ \frac{\partial \phi}{\partial n} + k\phi &= 0 \text{ on } C_f\end{aligned}$$

together with a proper radiation condition. Here  $\partial/\partial n$  is the normal derivative directed into  $D_+$  and  $V$  is a given function. In the following, we present three different methods for solving the floating body problem. The first two methods are based on the well-known John's Green function  $\gamma(p, q)$  [1] while the last one is based on a simplified Green's function [6]

$$\gamma_0(p, q) = \frac{1}{2\pi|p-q|} - \frac{1}{2\pi|p-q_1|}$$

where  $q_1$  is the reflection of  $q$  with respect to the bottom  $C_B$ . Each of these methods leads to a uniquely solvable boundary integral equation.

We begin with the representation:

$$\phi(p) = -\frac{1}{2} \int_{C_0} \left\{ \phi(q) \frac{\partial \gamma(p, q)}{\partial n_q} - \gamma(p, q) \frac{\partial \phi}{\partial n_q} \right\} ds_q$$

for  $p \in D_+ \cup C_f' \cup C_B$ , where  $C_f'$  denote points on  $C_f$  such that the boundary points between  $C_f$  and  $C_w$  are not included. Now if we extend  $\phi(p)$  on  $C_0$  to  $\Phi(p)$  on  $C_0 \cup C_w$  such that  $\Phi(p)|_{C_0} := \phi(p)$ ,  $p \in C_0$ , then it suffices to solve the boundary integral equation for  $\Phi$ :

$$\Phi(p) + \int_{C_0 \cup C_w} \beta(q) \Phi(q) \frac{\partial \gamma(p, q)}{\partial n_q} ds_q = \int_{C_0} V(q) \gamma(p, q) ds_q$$

for  $p \in C_0' \cup C_w'$ , where  $\beta(q) = \begin{cases} 1 & \text{for } q \in C_0 \\ 1/2 & \text{for } q \in C_w \end{cases}$ . Here again the primed notation is used to exclude the boundary points. That this equation is free of irregular frequencies is shown in [4].

A second method of obtaining an integral equation with, at most, one solution which does not involve extending the domain of the integral operator is patterned after the method used in acoustic scattering problems by Burton and Miller [9]. Here we solve the boundary integral equation of the form:

$$\begin{aligned} \phi(p) + \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p, q) ds_q + \eta \frac{\partial}{\partial n_p} \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p, q) ds_q \\ = \int_{C_0} V(q) \gamma(p, q) ds_q + \eta \left\{ \int_{C_0} V(q) \frac{\partial \gamma}{\partial n_p}(p, q) ds_q - V(p) \right\} \end{aligned}$$

for  $p \in C_0$ , when  $\eta$  is a suitable constant. Again one can show that this integral equation has, at most, one solution if  $\text{Im } \eta \neq 0$ , see [4] and [5].

For the final method, we consider the representation:

$$\begin{aligned} \phi(p) = -\frac{1}{2} \int_{C_0} \phi(q) \frac{\partial \gamma_0}{\partial n_q}(p, q) ds_q - \frac{1}{2} \int_{C_f} \phi(q) \left[ \frac{\partial \gamma_0}{\partial n_q} + k \gamma_0(p, q) \right] ds_q \\ + \frac{1}{2} \int_{C_0} V(q) \gamma_0(p, q) ds_q \quad \text{for } p \in D_+ \cup C_B. \end{aligned}$$

Here in contrast to the first two methods, the simplified Green function  $\gamma_0$  does not satisfy the homogeneous boundary condition on the free surface. As a consequence, we obtain an integral equation defined over both the wetted surface of the ship hull and the free surface. In fact, this leads to the integral equation:

$$\begin{aligned} \phi(p) + \int_{C_0} \phi(q) \frac{\partial \gamma_0}{\partial n_q}(p, q) ds_q + \int_{C_f} \phi(q) \left[ \frac{\partial \gamma_0}{\partial n_q}(p, q) + k \gamma_0(p, q) \right] ds_q \\ = \int_{C_0} \gamma_0(p, q) V(q) ds_q \quad \text{for } p \in C_0' \cup C_f', \end{aligned}$$

It was shown in [6], under suitable conditions, this integral equation has, at most, one solution.

### 3. The Iteration Method

All of the integral equations of the previous section are of the form

$$(I - A)u = f$$

where  $f$  is a known function and  $A$  in general is a non-selfadjoint, noncompact operator. Introduce the operator  $L$  by  $L := I - A$  where  $L$  maps a Banach space into itself. The appropriate function space depends on the particular choice of  $A$ .

Recall that the spectrum of  $L$ ,  $\sigma(L)$ , consists of all complex numbers  $\lambda$  for which  $(\lambda I - L)^{-1}$  does not exist, is unbounded, or the range of  $\lambda I - L$  is not dense in the Banach space while the spectral radius of  $L$ ,  $r(L)$ , is defined as

$$r(L) = \sup_{\lambda \in \sigma(L)} \{ |\lambda| \}.$$

The acceleration or relaxation parameter  $\alpha$  may be introduced by observing that if  $\alpha \neq 0$

$$Lu = f \iff u - (I - \alpha L)u = \alpha f$$

or, introducing the operator  $B_\alpha$  by

$$B_\alpha := I - \alpha L$$

the equation to be solved takes the form

$$(I - B_\alpha)u = \alpha f.$$

The underlying idea is to choose  $\alpha$  so that  $r(B_\alpha) < 1$  in which case Picard iteration converges, that is,

$$u = \lim_{n \rightarrow \infty} u_n$$

where  $u_0$  is arbitrary and

$$u_{n+1} = B_\alpha u_n + \alpha f$$

The question of whether it is possible to choose  $\alpha$  to guarantee convergence of the sequence of iterates (in the norm of the underlying function space) depends on  $\sigma(L)$ . Using the complex form of  $\lambda$  and of  $\alpha$ ,  $\lambda = |\lambda|e^{i \arg \lambda}$ ,  $\alpha = |\alpha|e^{i \arg \alpha}$ ,  $-\pi < \arg \lambda, \arg \alpha \leq \pi$ , define

$$\epsilon = \inf_{\lambda \in \sigma(L)} \{ |\lambda| \}$$

$$\theta = \inf_{\lambda \in \sigma(L)} \{ \arg \lambda \}$$

$$\theta = \sup_{\lambda \in \sigma(L)} \{ \arg \lambda \},$$

$$\theta_0 = \pi - \theta + \theta.$$

Then sufficient conditions for convergence are given in [10] as follows: if  $\epsilon > 0$ ,  $0 < \theta_0 \leq \pi$ ,  $\theta_1$  is any angle in  $(0, \theta_0)$  and  $\alpha$  is chosen such that

$$0 < |\alpha| < \frac{\sin \theta}{r(L)} \quad \text{and} \quad -\frac{\pi}{2} + \frac{\theta_1}{2} - \theta < \arg \alpha < \frac{\pi}{2} - \frac{\theta_1}{2} - \bullet$$

then  $r(B_\alpha) < 1$  and the iterative solution converges.

In order to verify that conditions for convergence are satisfied and that  $\alpha$  is appropriately chosen it is necessary to have considerable information about the spectrum of  $L$  (or equivalently  $\sigma(A)$ ). In general this information is not easily available however a practical and easily computed choice of  $\alpha$  has been suggested in [10] which has proven extremely effective in a number of examples. Specifically choose  $u_0 = 0$  and define  $\alpha$  to minimize

$$\|Lu_1 - f\| = \|\alpha Lf - f\|.$$

When the underlying space is in fact a Hilbert space with inner product  $(\cdot, \cdot)$ , then  $\alpha$  is found explicitly to be

$$\alpha = \frac{(f, Lf)}{\|Lf\|^2}.$$

Using this choice of  $\alpha$  with the  $L_2$  inner product it is necessary on the one hand to show that the conditions for convergence of the iterative procedure are met and on the other hand to demonstrate the practical utility in explicit numerical examples. These questions will be discussed for the integral equations given in section 2.

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**Papanikolaou:** I understand that a prerequisite for an iterative solution method to work properly for all frequencies is that the integral equation is free of irregular frequencies. This was pointed out in the late sixties in a paper by Milgram *et al*. Since I do not know of any 3-D integral-equation-methods without difficulties at some frequencies of oscillation, I question the validity of the iterative scheme when computations are performed over the whole frequency range of interest.

**Hsiao & Kleinman:** The existence of integral-equation formulations which are totally free of irregular frequencies is well established and we have referred to some of them in the abstract. [Another, which in many ways is my favorite, is based on the use of a modified Green function using the wave free potentials of Ursell.] How well these various methods perform numerically is a question I cannot answer, but an indication of the efficacy of one such method is given in the presentation by Lee.

**Wehausen:** The existence of a positive lower bound  $m$  seems like a particularly restrictive requirement and a difficult one to establish. Once one knows that  $m > 0$  one knows that the operator is invertible and takes open sets into open sets. If one can establish that, it does not really surprise me that one can find efficient iteration schemes.

**Hsiao & Kleinman:** The requirement of the operator being bounded below is indeed stringent but not totally impossible to ascertain without complete knowledge of the spectrum. Recall that the operator  $L = I - A$ . The requirement that  $L$  be invertible (*i.e.* that there are no irregular frequencies) means that 0 is not in the spectrum of  $L$ . If the operator  $A$  were compact then 0 would be the only possible accumulation point for points in the spectrum of  $A$  which means that  $I$  would be the only possible accumulation point for the spectrum of  $L$ . Since by removing irregular frequencies we ensure that 0 is not in the spectrum, and 0 is not an accumulation point, then  $L$  is bounded below. Of course this argument depends on  $A$  being compact, which is not generally the case (there are exceptions, perhaps when the hull intersects the free surface *normally* is one) but it does lend credence to the plausibility of  $L$  being bounded below.

**Tuck:** The same operation count would be achieved if one put any diagonal matrix on the LHS instead of a multiple of the unit matrix. Could this not be more efficient?

**Hsiao & Kleinman:** We think so. One would have to develop some sort of theoretical indication (if not proof) of convergence, but there are a variety of modifications of matrix equations which have been studied. We have only indicated one which can be studied in operation form before discretization to arrive at a linear algebraic system.

**Lee:** Is the practical formula for  $\alpha$  applicable to other iterative methods, for example Gauss-Siedel?

**Hsiao & Kleinman:** Introduction of the empirically determined  $\alpha$  can be interpreted as a preconditioning, after which we think only some sort of relaxation method is useful. Either the simple Picard iteration or the SOR (which is slightly different) could be employed. A comparison of results would be interesting in those cases. If standard Gauss-Siedel is used with the matrix,  $I - B\alpha$ , then the introduction of the relaxation or acceleration parameter has no effect.

**Sclavounos:** Powerful "numerical" iterative methods have appeared which seem to produce rapidly convergent solutions often without any knowledge of the integral-equation spectrum. Would you

comment on the importance of designing an iterative method tailored to take advantage of the spectrum of a given integral equation relative to using general methods, for example the conjugate-gradient methods. Is it likely that a certain class of general methods are expected to be successful for operators of the form  $I + A$  where  $A$  is compact or bounded?

**Hsiao & Kleinman:** It is not quite true that iterative methods such as conjugate-gradient do not require any knowledge of the spectrum. To prove convergence of CG one uses the fact that the operator is bounded below (or that zero is not an eigenvalue). But having said that, there is a good argument for using CG and not bothering about other methods. Our interest is twofold: one, in trying to see if there is some way to partially implement an iterative method or achieve a form which would enhance the rapidity of convergence analytically before discretization; and two, to propose a useful iteration scheme which is cheaper and easier to implement than CG. There are matrix methods which improve convergence rates by employing detailed knowledge of the spectrum provided the spectrum is known to be contained in some region, e.g. an ellipse, but the cost of obtaining this knowledge for the integral operators or matrices that arise in free-surface hydrodynamics would be prohibitive in practice. For theoretical studies in special cases, however, this could be done and might indeed be very interesting.