A TREATMENT OF THE SECOND ORDER DERIVATIVES FOR A SHIP ADVANCING IN WAVES

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1. Introduction

It is well known that in the linearized potential flow analysis of a ship advancing at constant forward speed in regular waves a major difficulty is that the unsteady potential contains the second order derivatives of the steady potential due to forward speed[2]. Since the steady potential itself, which is the solution of the Neumann-Kelvin problem, needs complex numerical modelling, high accuracy of its second order derivatives is very difficult to obtain. However, it has been shown[4] that this difficulty can be easily avoided by the use of the localized finite element method in the near field combined with a boundary integral equation in the far field[1], instead of the more common source distribution method. The results given here suggest that only the first order derivatives are then needed. To demonstrate the principle, we provide two examples without free surface effects.

2. Governing equation

Besides satisfying the Laplace equation, the steady potential $\bar{\phi}$ and the six components ϕ_j of the unsteady potential satisfy the following conditions on the body surface S_0[2]

$$\frac{\partial\bar{\phi}}{\partial n} = n_1 \;;\qquad \frac{\partial\phi_j}{\partial n} = i\omega n_j + U m_j \tag{1}$$

where ω is the body oscillation frequency and U is forward speed; $\mathbf{n}$ is the normal of the body surface with the components

$$\mathbf{n} = (n_1, n_2, n_3) \;;\qquad \mathbf{X}\times\mathbf{n} = (n_4, n_5, n_6) \tag{2a}$$

$$-(\mathbf{n}.\nabla)\mathbf{W} = U(m_1, m_2, m_3) \;;\qquad -(\mathbf{n}.\nabla)(\mathbf{X}\times\mathbf{W}) = U(m_4, m_5, m_6) \tag{2b}$$

$$\mathbf{W} = U\nabla(\bar{\phi} - x) \tag{2c}$$

For the body in an unbounded fluid domain, the potential also satisfies the radiation condition which requires that the potential tends to zero at infinity.

3. Coupled finite element method

Instead of satisfying the Laplace equation in the whole fluid domain, the finite element method imposes the Laplace equation in a uniform sense. We have[4]

$$\iiint_{R_1} \nabla^2\phi\psi d\sigma = 0 \tag{3}$$

where R_1 is a fluid domain surrounding the body surface, and ψ is an appropriately chosen weight function. From Green's identity, (3) becomes

$$\iiint_{R_1} \nabla\phi\nabla\psi d\sigma - \iint_{S_j} \frac{\partial\phi}{\partial n}\psi dS = \iint_{S_0} \frac{\partial\phi}{\partial n}\psi dS \tag{4}$$

where S_j is the outer boundary of R_1. In (4), $\frac{\partial\phi}{\partial n}$ on the right hand side is known from the body surface condition; $\frac{\partial\phi}{\partial n}$ on the left is to be determined by an integral equation in the outer domain R_2. We obtain[1,4]

$$\phi = \frac{1}{\alpha}\iint_{S_j}[\phi\frac{\partial G}{\partial n} - \frac{\partial\phi}{\partial n}G]dS \tag{5}$$

where α is the subtended angle and the Green function is the Rankine source. Combining (4) and (5), we can eliminate $\frac{\partial\phi}{\partial n}$ from (4) and obtain an equation for ϕ.

As discussed in the introduction, one of the main difficulties is the second order derivatives on the right hand side of (4). But since in the present method they appear in an integral form, it is possible to reduce the order of these derivatives. Taking ϕ_1 in two dimension as an example, we have

$$U\int_{S_0} m_1 N_j dS = -U\int_{S_0}[N_j\frac{\partial^2\bar{\phi}}{\partial x^2}n_1 + N_j\frac{\partial^2\bar{\phi}}{\partial x\partial z}n_3]dS$$

$$= -U\int_{S_0}\{N_j\frac{\partial^2\bar{\phi}}{\partial x^2}n_1 + \frac{\partial}{\partial x}[N_j\frac{\partial\bar{\phi}}{\partial z}]n_3 - \frac{\partial N_j}{\partial x}\frac{\partial\bar{\phi}}{\partial z}n_3\}dS$$

where we have taken ψ as the finite element shape function N_j, since its choice is quite arbitrary. Using $\nabla^2\bar{\phi}=0$, $W.n=0$ on S_0, $dx=n_3 dS$ and $dz=-n_1 dS$, and noticing that the second integration is zero, we obtain

$$U\int_{S_0} m_1 N_j dS = U\int_{S_0}[-N_j\frac{\partial^2\bar{\phi}}{\partial z^2}]dz - U\int_{S_0}\frac{\partial N_j}{\partial x}\frac{\partial}{\partial x}(\bar{\phi}-x)n_1 dS$$

$$= -U\int_{S_0}[\frac{\partial N_j}{\partial z}\frac{\partial\bar{\phi}}{\partial z} + \frac{\partial N_j}{\partial x}\frac{\partial}{\partial x}(\bar{\phi}-x)]n_1 dS = -\int_{S_0} W\nabla N_j n_1 dS \tag{6}$$

One can thus prove for both the two and three dimensional cases[3]

$$U\int_{S_0} m_i N_j dS = -\int_{S_0} W\nabla N_j n_i dS \tag{7}$$

4. Numerical examples

We give examples of a circular cylinder and a sphere, to demonstrate how equation (7) works for the two and three dimensional problems. We define ψ_j as the solution of the unsteady potential without forward speed and satisfying the body surface condition

$$\frac{\partial\psi_j}{\partial n} = n_j \tag{8}$$

From (1), we immediately have for j=1,2,3

$$\phi_j = i\omega\psi_j - U\frac{\partial\psi_1}{\partial x_j} \tag{9}$$

where $(x_1,x_2,x_3)=(x,y,z)$. From the solution for ψ_i in the polar coordinate system $(x,y)=(r\cos\theta, r\sin\theta)$, we have for a circular cylinder

$$\phi_1 = -i\omega\frac{a^2}{r}\cos\theta - \frac{Ua^2}{r^2}\cos 2\theta; \quad \phi_3 = -i\omega\frac{a^2}{r}\sin\theta - \frac{Ua^2}{r^2}\sin 2\theta \tag{10}$$

From the potentials we obtain the added masses μ_{ij} as[4]

$$\mu_{11} = \mu_{33} = \rho\pi(a^2 + \frac{2U^2}{\omega^2}); \quad \mu_{13} = \mu_{31} = 0 \tag{11}$$

Table 1 compares the analytic solution and the numerical results from 12 elements using 8 point quadratic shape function. It can be seen that they are in very good agreement.

Similarly, we obtain the solution for the sphere

$$\phi_1 = -\frac{i\omega}{2}\frac{a^3}{r^2}\cos\theta - \frac{U}{2}\frac{a^3}{r^3}(3\cos^2\theta - 1) \tag{12}$$

in spherical coordinate system $(x,y,z)=(r\cos\theta, r\sin\theta\cos\alpha, r\sin\theta\sin\alpha)$ with corresponding results for ϕ_2 and ϕ_3[4]; and the added masses are[4]

$$\mu_{11} = \rho\pi(\frac{2}{3}a^3 + \frac{12}{5}\frac{U^2}{\omega^2}a); \quad \mu_{22} = \mu_{33} = \rho\pi(\frac{2}{3}a^3 + \frac{9}{5}\frac{U^2}{\omega^2}a) \tag{13}$$

with the cross terms being zero. Table 2 compares the comparison of the analytic solution and the numerical results for the sphere based on 54 elements and 20 point quadratic shape functions. It can be seen that they are in fairly good agreement. It is believed that the accuracy of the numerical results can be further improved by using a finer mesh of more elements.

5. Concluding remarks

Equation (7) was originally obtained by Ogilvie and Tuck. This result was thought to be useful to avoid the computation of the derivatives of the unsteady potential in the calculation of the hydrodynamic coefficients. We have noticed however that the reversed application of this equation is particularly useful in the solution of the potential unsing the coupled finite element method. We have found that for an oscillating body at forward speed below a free surface this approach is also very successful[4]. For a surface ship, equation (7) will contain a term including a line integral of the shape function. This will not offer numerical difficulty, since the calculation of the mean drift force has a similar line integral, and results using the shape function have been found quite satisfactory. Finally, it is interesting to point out that the influence of $\bar{\phi}$ on ϕ_i can not be neglected in general, as we can see from equations (10) to (13).

References

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Table 1 Added mass of a circular cylinder for $U=0.4\sqrt{ga}$

$\frac{\omega^2 a}{g}$	numerical $\frac{\mu_{11}}{\rho\pi a^2}$	numerical $\frac{\mu_{33}}{\rho\pi a^2}$	analytic
0.1	4.1964	4.1175	4.2000
0.2	2.5969	2.5575	2.6000
0.3	2.0638	2.0375	2.0667
0.4	1.7972	1.7775	1.8000
0.5	1.6373	1.6215	1.6400
0.6	1.5306	1.5175	1.5333
0.7	1.4545	1.4432	1.4571
0.8	1.3973	1.3875	1.4000
0.9	1.3529	1.3442	1.3555
1.0	1.3174	1.3095	1.3200

Table 2 Added masses of a sphere for $U=0.4\sqrt{ga}$

$\frac{\omega^2 a}{g}$	$\frac{\mu_{11}}{4/3\pi\rho a^3}$		$\frac{\mu_{22}}{4/3\pi\rho a^3}$	
	numerical	analytic	numerical	analytic
0.1	3.3008	3.3800	2.3768	2.6600
0.2	1.8978	1.9400	1.4357	1.5800
0.3	1.4300	1.4600	1.1220	1.2200
0.4	1.1962	1.2200	0.9652	1.0400
0.5	1.0559	1.0760	0.8711	0.9320
0.6	0.9623	0.9800	0.8083	0.8600
0.7	0.8955	0.9114	0.7635	0.8086
0.8	0.8454	0.8600	0.7299	0.7700
0.9	0.8064	0.8200	0.7038	0.7400
1.0	0.7752	0.7880	0.6828	0.7160

Discussion

Palm: You point out in your lecture that a major difficulty in your problem is that the unsteady potential contains the second order derivatives of the steady potential due to forward speed. I would like to point out that in two-dimensional problems these second-order derivatives may be expressed by first order derivatives along the body surface. This is applied for a submerged circular cylinder in Grue and Palm (J.Fluid Mech. (1985),151, 257-278).