RECENT ADVANCES IN UNIFIED THEORY

by

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ABSTRACT

Waves can be generated by elongated bodies in a variety of physical contexts. A ship forced to oscillate on a free surface, a wavequide vibrating in an acoustic medium and a high-aspect ratio lifting surface shedding a wave-like vortex sheet when forced in time-harmonic motion, offer three diverse examples. In all three problems a wavelength can be identified which may be comparable to the body transverse or its longitudinal dimension. Slender-body approximations for the first two problems for wavelengths ranging from the transverse geometry scale to the infinite wavelength limit have been obtained by using the techniques of unified theory developed by Newman (1978). The present talk studies the latter of the three problems within the same framework and presents a solution also uniformly valid for wake-wavelengths that range from the chord to the infinite wavelength limit which is Prandtl's lifting-line theory. Possible applications of interest in surface-wave body interactions are in the design of hydrofoil-boats advancing in waves or in the evaluation of the lift and drag experienced by sailboat keels.

A plane high-aspect ratio wing advancing at a forward speed U and forced to oscillate at a frequency ω sheds a wake on which the bound-vorticity strength oscillates in the direction of advance with a characteristic wavelength $\lambda = 2\pi U/\omega$. In light of the equivalence of vortex and dipole representations, at distances large compared to the wing span the flow can be approximated by a distribution of dipoles normal to the mean wake position, assumed to be plane, with moment slowly varying in the direction of the span and sinusoidal with wavelength λ in the direction of the forward motion.

Information about the flow variation near the wing axis (assumed to coincide with the x-axis), is revealed by expanding the far-field representation outlined in the preceding paragraph for small values of the transverse radial distance R from the wing axis. Uniformity of this expansion is first desired with respect to the x-coordinate. In unified theory, this is accomplished via a "factorization" of the x-dependence of the far-field solution by Fourier transforming it with respect the the x-coordinate. This operation effectively reduces the dimension of the problem, since the Fourier transform of the velocity potential satisfies a two-dimensional modified Helmholtz equation on planes normal to the x-axis which depends on the real Fourier wavenumber k.

The derivation of a small-R expansion of the transformed potential is no trivial exercise in general. For a distribution of free-surface wave sources on the axis of a ship forced to oscillate in a time-harmonic manner, Ursell (1962) derived a convergent ascending series expansion in terms of modified Bessel functions of the first and second kind which are known to be solutions of the modified Helmholtz equation. For a body vibrating in an acoustic medium the corresponding expansion is easier to derive. Ultimately these expansions are transformed back into the physical x-space. Uniformity with respect to the x-coordinate is ensured by requiring that the Fourier wavenumber k is a quantity of O(1) for small values of R. Uniformity with respect to the physical wavenumber $\nu = \omega/U$ is more subtle. It rests with our ability to ensure that the series expansion is convergent.

Conventional matched-asymptotic-expansions theory for flows around slender bodies derives far- and near-field expansions by idendifying the disparity of the two geometrical length scales and relating the wavelength in an asymptotic sense to one of them. This methodology permits a systematic construction of a sequence of far and near solutions in increasing powers of the small parameter ε which measures the ratio of the two geometry scales, but inevitably restricts the magnitude of the wavenumber. Their compatibilty is enforced in an appropriately defined intermediate regime. A comprehensive account of this technique for a wide class of problems in fluid mechanics is given by Van Dyke (1964).

Unified theory adopts a different philosophy. The approximations are derived to leading-order in ε , but the wavelength is assumed to be asymptotically unconstrained. It turns out that in deriving the near-field expansion of the far-field solution the latter requirement can be honored only if the small-R expansion of the far-field solution is a convergent series which is not truncated after a finite number of terms. This can be justified as follows. The asymptotic mathcing of the far- and near-field solutions is carried out in a "matching" region the transverse distance of which form the origin R=0 decreases in the asymptotic limit $\varepsilon \to 0$. It is at such "small" values of R that the expansion of the far-field solution solution will be utilized. It would thus appear sufficient to truncate it after a finite number of terms. The variation of the inner expansion over distances comparable to the radius R_m of the matching region depends on the relative order of the wavelength λ to R_m. If $R_m \ll \lambda$ as $\varepsilon \to 0$, then a finite number of terms would be sufficient in the inner expansion. If on the other hand the asymptotic theory is expected to be uniformly valid up to wavelengths comparable to the body transverse dimension (for example the beam of a ship, here the chord of the wing), then in the limit $\varepsilon \to 0$, $R_m \gg \lambda$ since the mathcing region is "far" from the near field which is of comparable extent to the transverse geometry scale. In the latter case, all terms are needed in the ascending series expansion of the far-field flow in order to describe the rapid flow variation over distances which range from R=0 to R_m.

If all terms are kept in the series it is of course impossible to approximate the far-field solution near the body axis. Progress is possible by requiring that the leading-order term in the inner approximation of the far-field solution is a solution of the two-dimensional Laplace equation which is expected to be relevant in the near field. The observation that for a zero value of the Fourier wavenumber k the modified Helmholtz equation reduces to the two-dimensional Laplace equation suggests that the leading-order term in the inner expansion of the outer solution must be the k=0 limit of its ascending series expansion. This is a function of the transverse (y,z) coordinates and consists of an infinite number of terms, needed to preserve uniformity with respect to the physical wavenumber. The next-order term depends on k and consequently accounts for hydrodynamic interactions in the direction of the body axis, since when transformed back in the physical x-space, k-dependent terms are synonymous to variation in the x-direction. Formally, it can be obtained as the first term in the Taylor series expansion for small R of the difference of the exact ascending series minus its k=0 limit.

A convergent ascending series in R has been derived for the far-field flow due to a dipole distribution on the wake of the lifting surface in a form that formally resembles to the series derived by Ursell for the ship problem. Their k=0 limit is the exact velocity potential due to a two-dimensional point vortex at R=0 forced to oscillate at a frequency ω in the presence of a uniform stream U. The next-order term which accounts for the spanwise variation of the flow turns out to be the product of a function which depends on k, times the vertical spatial coordinate y. When transformed back in the x-space, the resulting velocity potential represents an induced vertical velocity ("downwash") which varies gradually along the spanwise direction. For large values of the frequency of oscillation it tends to zero and in the limit of infinitely long waves, or zero frequency, it is identical to the downwash presdicted by the Prandtl lifting-line theroy.

The two-term inner expansion of the far-field velocity potential supplies sufficient information for the formulation of the near-field flow. The downwash induced in the near field by the far-field flow leads to an effective change in the local angle of attack at each transverse section of the lifting surface. The local circulation is thus obtained by solving a sequence of purely two-dimensional oscillatory lifting problems at angles of attack which vary in the direction of the span. The result is an integro-differential equation for the spanwise distribution of circulation which is uniformly valid for wavelengths which range from the chord to the infinite-wavelength limit. In the limit of high frequency of oscillation the induced downwash tends to zero and the problem reduces to the solution of a sequence of non-intercating two-dimensional problems. In the opposite limit of infinitely low frequency the integral equation reduces to that obtained by Prandtl in his lifting-line theory.

Computations are presented of the lift and mean drag obtained by the present theory over the whole frequency and aspect-ratio range. The integro-differential equation is solved by approximating the distribution of circulation along the span in a sine series, as is often done in the solution of the lifting-line equation. Comparisons are made with predictions obtained from an independent three-dimensional lifting-surface computer program which has been developed for lifting surfaces of general planform and aspect ratio. The problem is solved in the time domain with the lifting surface forced in a time-harmonic vertical oscillation initiated at t=0. The transient lift and drag are frequency analysed for a sufficiently large duration of forward advance, which permits the evaluation of their frequency-domain modulus and phase.

REFERENCES

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Discussion

Schwartz: Lifting-line theory assumes that the induced velocity on the wake itself is of higher order. When oscillations occur is this still a good assumption? Does your numerical theory account for this effect?

Sclavounos: The wake in the present study is convected by the velocity U only, not by the perturbation velocity. We do not account for wake roll-up. The numerical solution by Lee was done in the time domain. The wake vorticity induced a normal velocity on the foil, but not on itself.

Beck: So there is no roll-up?

Sclavounos: The wake vorticity was assumed to be in a plane. The roll-up was neglected.

T. Wu: Recently Daniel Weihs and Joseph Katz considered the problem of vortex shedding and the effects of the velocity of their transport. They dealt with two typical cases by assuming that the shed vorticity is transported with the free-stream velocity or that it is transported with the induced velocity. Little difference on the net result of the lift was observed. My question is how many elements were used in your vortex lattice method?

Sclavounos: The vortex lattice method used a 4 X 8 lattice, 8 chordwise, 4 along half the span.

Tuck: You quoted some previous publications that possess singularities in some limits. Can you identify how unified theory improves upon these theories by eliminating the terms that blow up?

Sclavounos: There are two principal differences between unified theory and previous theories:

i) We start with a distribution of dipoles on the wake as opposed to an equivalent distribution over the foil which is relevant when the acceleration potential is used.

ii) The leading-order term in the inner expansion of the outer solution is taken to be the exact velocity potential of a two-dimensional time-harmonic vortex. Previous low-frequency theories include only a few terms in its Taylor series expansion for small frequencies, thus preventing their uniform validity across the frequency spectrum.